Math Challengers®

Friday, January 29th, 2016
★ Mock Competition ★
Blitz Round
Answer Key & Full Solutions Manual

Form Code

A		F		0		5	
B	■	G		1	■	6	
C		H		2		7	
D		I		3		8	
E		J		4		9	

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1. 36

1. $1 + 11 = 12$. $3 + 9 = 12$. $5 + 7 = 12$. $12 \times 3 = 36$. Manual adding will give the same result.

2. $\frac{1}{36}$

2. The probability of rolling a 1 is $\frac{1}{6}$. The probability of doing so on both dice is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

3. $\frac{3}{2}$

3. $1@(1@1) = 1@(1 + \frac{1}{1}) = 1@2 = 1 + \frac{1}{2} = \frac{3}{2}$

4. 4

4. The most number of balls we can pick without choosing two of the same colour is three: one blue ball, one red ball, and one green ball. After that, any ball chosen will have the same colour as a previously chosen ball.

5. $\frac{1}{4}$

5. The probability of flipping a head on the coin is $\frac{1}{2}$. The probability of rolling a prime number on the die (2, 3, or 5) is $\frac{3}{6} = \frac{1}{2}$. The probability of achieving both outcomes is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

6. 8

6. The circumference of a circle is $2\pi r$, with r being the radius. Therefore the radius of our circle is 2 ($2\pi r = 4\pi \to r = 2$). The inscribed square will have a diagonal passing through the center of the circle so the diagonal must have a length twice this radius, or 4. If the square has side length x, applying the Pythagorean Theorem gives $x^2 + x^2 = 4^2$. Since we need the area of this square, we want the value of x^2, which is 8 by the previous equation.

7. 0

7. Note that 10^3 is divisible by 200 ($10^3 = 200 \times 5$). So 10^4 must also be divisible by 200, and in fact, all integral powers of 10 with the exponent greater than 3 is divisible by 200. Therefore the desired remainder is zero.

8. 2025

8. A number is divisible by 9 if and only if the sum of its digits is divisible by 9. Using this property, we can see that 2016 is a multiple of 9, because 2+0+1+6=9 which is divisible by 9. We then simply add 9 to 2016 obtain 2025, the smallest number larger than 2016 that's divisible by 9.

9. 3

9. The Triangle Inequality Theorem states that in any triangle with sides a, b, and c, $a + b > c$, $a + c > b$, and $b + c > a$. The smallest value integer value for x that satisfies these equations is 3: $6 + 3 > 8$.

10. 12

10. There are only four sets of positive integers (up to ordering) $\{a, b, c\}$ that satisfy $a + b + c = 7$. These are $\{1, 1, 5\}$ with product 5, $\{1, 2, 4\}$ with product 8, $\{1, 3, 3\}$ with product 9, and $\{2, 2, 3\}$ with product 12. Therefore the largest possible value of abc is 12.

11. 54

11. Let the number of students who wore red and white be a, the number of students who wore green and white be b, the number of students who wore red and green be c, the number of students who wore all three colors be d, and the number of students who wore none of the three colors be e. Then, we have, by drawing a suitable Venn Diagram, the equation $55+48+21-a-b-c+d = 70-e$. This simplifies to $54 = a+b+c-d-e$. To maximize the number of students that wore exactly two of the three colors, we want to maximize $a+b+c$ while minimizing d and e. Letting $d = 0$ and $e = 0$ gives us $54 = a + b + c$. This is achievable if the following occurs:

- 10 students wore only red, 4 students wore only green, and 2 students wore only white
- 35 students wore red and green, but not white, 10 students wore red and white, but not green, and 9 students wore green and white, but not red
- Every student wore at least one of the three colors and no student wore all three colors

Of course, this is just one example that satisfies the conditions in the problem. We leave it to the interested reader to determine all other possible examples of the shirt color distribution for the students at this party.

12. \$69.03

12. $\$19.89 \times 2 + \$9.75 \times 3 = \$39.78 + \$29.25 = \$69.03$

13. 6

13. We can just list them out; the positive integers less than 100 with a units digit of 7 are 7, 17, 27, 37, 47, 57, 67, 77, 87, and 97. 87, 57, and 27 are divisible by 3, so we eliminate them from the pool. 77 is divisible by 7 and 11, so we take it away as well. All other integers can be tested for their divisibility to reveal that they are prime. Thus, we are left with 6 primes.

14. 5

14. If there are x red balls in the bag originally, the probability of drawing a red ball is $\frac{x}{10}$. The probability of drawing a red ball twice in a row is $\frac{x}{10} \times \frac{x}{10} = \frac{x^2}{100}$ which gives $\frac{x^2}{100} = \frac{1}{4}$ according to the problem statement. Simplifying yields $x^2 = 25 \to x = 5$ as x must be non-negative.

15. $\frac{4028}{17}$

15. The sum of the internal angles of a 7-sided polygon (a heptagon) is $180(7-2) = 900$. The internal angles we have are

$$x+7, \quad 2x-25, \quad \frac{3x}{2}+7, \quad \frac{x}{2}+9, \quad \frac{x}{2}+42, \quad 1.5x, \quad 1.5x.$$

We know these numbers add up to 900, because the sum of the internal angles of a heptagon is 900. Adding all the angles up and then solving for x yields $x = \frac{1720}{17}$. Plugging x into the values in the set, the largest element is $2x - 25$, and the smallest element is $\frac{x}{2} + 9$. Adding the two values together, we get $\frac{4028}{17}$.

16. 85

16. Manipulating the given recurrence equation $t_n = t_{n-1} + t_{n-2}$ gives $t_{n-1} = t_n - t_{n-2}$. Plugging in $n = 9$ gives $t_8 = 54 - 23 = 31$. Then we have $t_{10} = t_9 + t_8 = 54 + 31 = 85$

17. 56.4

17. A rough sketch shows that the region bounded by the four lines is a quadrilateral in the fourth quadrant. We can name its vertices A, B, C, and D starting from the origin and going clockwise. This quadrilateral can be split into a trapezoid and a triangle (to make the desired area are easier to calculate) by drawing a horizontal line segment CE from C to point E on side AD such that $CE \perp AD$. The trapezoid's height is AE, its shorter base is AB, and its longer base is CE. The triangle's height is DE and its length is also CE.

The trapezoid's area can be calculated from $\frac{1}{2}\times$(shorter base+longer base)$\times$height.

- **Shorter base**:
  B is the x-intercept of the line $5x + 4y - 48 = 0$. For the x-intercept of a line, $y = 0$. Plugging that into $5x + 4y - 48 = 0$ gives us $x = \frac{48}{5}$. So segment AB is $\frac{48}{5}$ in length.
- **Longer base**:
  C is the intersection point of $5x + 4y - 48 = 0$ and $x - 3y - 21 = 0$. We can find its coordinates by combining the two equations and then solving the system of equations.

  $$x - 3y - 21 = 0 \to 5x - 15y - 105 = 0$$

  Subtracting the above equation on the right from $5x + 4y - 48 = 0$ gives $19y + 57 = 0$, or $y = -3$. This the the y-coordinate of C.
  Plugging y into either equation gives $x = 12$. This is the x-coordinate of C, which tells us that line segment $CE = 12$.
- **Height**:
  Since CE is a horizontal line, the length of the perpendicular from C to the x-axis is the same as the length of line segment AE. Since C's y-coordinate is -3, AE is 3.

Therefore the area of the trapezoid is $\frac{1}{2} \times (\frac{48}{5} + 12) \times 3 = 32.4$.

The triangle's area is $\frac{1}{2}\times$base$\times$height.

- **Base**:
  From before, $CE = 12$.
- **Height**:
  The line $x - 3y - 21 = 0$ has y-intercept $(0, -7)$, so line segment AD has length 7. Since AE has length 3, DE has length $7 - 3 = 4$.
  Therefore the area of the triangle is $\frac{1}{2} \times 12 \times 4 = 24$.

Adding the areas of the trapezoid and the triangle gives the area of $ABCD$, which is $32.4 + 24 = 56.4$.

18. 72

18. The non-negative integers that have at least one digit of 2 are: 2, 12, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 32, 42, 52, 62, 72. . .. The 2 in 72 is the 18th 2 written down, which gives us our answer.

19. 108

19. If you put two of these triangular prisms together, you get a rectangular prism with a square as two of its faces. These square faces have a side length of 6 (the square diagonals of lengths $6\sqrt{2}$ as given in the problem statement). T rectangular prism's third dimension is 3 (given as the height of the triangular prism). If you put two of these rectangular prisms together, you get a cube with side length 6. $(18 \times 18 \times 18) \div (6 \times 6 \times 6) = 27$. Since each of the cubes with side length 6 is made of 4 triangular prisms, we multiply 27 by 4 to get our answer: 108.

20. 204

20. From the problem, we get two equations; $n\pi = 4 \cdot 2\pi r$ and $n\pi = 3 \cdot 2\pi(r + 17)$. Solving the equations, we get $r = 51$ and $n = 408$. A square with side length $r = 51$ has a perimeter of 204, so $3 \cdot 204 = y + 408$ according to the problem statement. Solving for y, we get $y = 204$. So the rope has to be extended by 204 meters.

21. 1

21. According to the laws of modular arithmetic, $a \equiv b \pmod{c}$ implies $a^n \equiv b^n \pmod{c}$ for appropriate integers a, b, c, and n. We notice that $2015 \equiv -1 \pmod{24}$, so $2015^{1000} \equiv (-1)^{1000} \pmod{24}$. So the desired remainder is 1.

22. -1

22. Plugging $x = 0$ into the left hand side of the equation, we get 8. We also notice that since all of the coefficients of the polynomial on the left hand side are positive, increasing the value of x above $x = 0$ will only increase the left hand side of the equation. Therefore, there are no non-negative real solutions to the equation. Now, plugging $x = -1$ into the left hand side of the equation gives $(-1)^7+3(-1)^6+7(-1)^5+4(-1)^4+8(-1)^3+6(-1)^2+(-1)+8 = 4$. Plugging $x = -2$ into the left hand side of the equation gives $(-2)^7+3(-2)^6+7(-2)^5+4(-2)^4+8(-2)^3+6(-2)^2+(-2)+8 = -130$. Hence, there must be exactly one real solution between $x = -2$ and $x = -1$ (the problem statement says there is only one real solution to the equation). To determine which integer this is closest to, we need to plug in $x = -1.5$ into the left hand side and see whether the value is positive or negative. So we have $(-1.5)^7 + 3(-1.5)^6 + 7(-1.5)^5 + 4(-1.5)^4 + 8(-1.5)^3 + 6(-1.5)^2 + (-1.5) + 8$.

To simplify this without a calculator, we pair up every two consecutive terms and factor out a 1.5 from the term with the higher power of 1.5, to get $(1.5)^7 - 6.5(1.5)^4 - 6(1.5)^2 + 6.5$. It's easy to see that $1.5^3 < 6.5$ so $(1.5)^7 - 6.5(1.5)^4 = 1.5^3(1.5)^4 - 6.5(1.5)^4 < 0$. Also, it's easy to see that $-6(1.5)^2 + 6.5 < 0$. Hence, $(1.5)^7 - 6.5(1.5)^4 - 6(1.5)^2 + 6.5 < 0$. Since plugging in $x = -1.5$ gives a negative value on the left hand side, and plugging in $x = -1$ gives a positive value on the left hand side, there must be a real root between $x = -1.5$ and $x = -1$. So the integer this root is closest to is $x = -1$.

23. 6

23. If the three digit integer contains the digits $a, b,$ and c, then we must have $abc = a + b + c$. Suppose without loss of generality that $a \geq b \geq c$. Then $a + b + c \leq 3a$. So $abc \leq 3a$. It's clear that a cannot be zero (if $a = 0$, then $abc = 0 = b + c$ so $b = 0$ and $c = 0$ which doesn't give us a three digit integer), so $bc \leq 3$. This gives us the solutions $b = c = 1$, $b = 2, c = 1$, and $b = 3, c = 1$ ($b, c \neq 0$ for the same reason that $a \neq 0$). If $b = c = 1$, then $a = a + 1 + 1$ which is absurd. If $b = 2, c = 1$, then $2a = a + 2 + 1$ which gives $a = 3$. If $b = 3, c = 1$, then $3a = a + 3 + 1$ which gives $a = 2$, but since $a \geq b$, this cannot happen either. So we must have $a = 3, b = 2, c = 1$. However, there are $3! = 6$ ways that we can arrange these three digits, which gives us our answer of 6. The interested reader can generalize this problem to positive integers with any number of digits.

24. 245

24. To get to point $(2,3)$, 3 ups and 2 rights are required along with either a left(L) and right(R) or (this is the exclusive or) an up(U) and down(D). The order that these moves go in does not matter, so we need to find the number of ways these moves can be permuted. The number of ways to rearrange the sequence of moves UUUURRD is $\frac{7!}{4!\cdot 2!} = 105$ and the number of ways to rearrange the sequence of moves UUURRRL is $\frac{7!}{3!\cdot 3!} = 140$. Our final answer is therefore $140 + 105 = 245$.

25. $\frac{61}{2}$

25. For the sake of simplicity, we will use the Shoelace Theorem to calculate our answer for this problem. The Shoelace Theorem can be used to calculate the area of a polygon given the coordinates of the polygon's vertices. The formula for the Shoelace Theorem requires us to first list the coordinates of the polygon in clockwise order: (a_1, b_1), $(a_2, b_2) \cdots (a_n, b_n)$. Then the desired area is: $|\frac{1}{2}[(a_1b_2 + a_2b_3 + \ldots + a_nb_1) - (b_1a_2 + b_2a_3 + \ldots + b_na_1)]|$. When you plug the given coordinates into the above formula, we get $|\frac{1}{2}[(3 \cdot 9 + 7 \cdot 11 + 2 \cdot 8 + 0 \cdot 3 + 1 \cdot 4) - (4 \cdot 7 + 9 \cdot 2 + 4 \cdot 0 + 8 \cdot 1 + 3 \cdot 3)]| = \frac{124-63}{2} = \frac{61}{2}$. If the reader is not familiar with the Shoelace Theorem, circumscribing a rectangle around this pentagon and then sequentially removing trapezoids and triangles from this rectangle is also a possible strategy to use to solve this problem.

26. $\frac{2}{9}$

26. If a is odd and b is odd, $ab - a + b =$odd$-$odd$+$odd$=$odd. If a is even and b is odd, then $ab - a + b =$even$-$even$+$odd$=$odd. If a is odd and b is even, then $ab - a + b =$even$-$odd$+$even$=$odd. If both a and b are even, then $ab - a + b =$even$-$even$+$even$=$even. Hence, to satisfy the condition that $ab - a + b$ is even, a and b must both be even. The probability of randomly choosing both a and b to be even is equal to the number of ways that both a and b are even divided by the total number of ways that 2 integers can be chosen from 10 integers: $\frac{\binom{5}{2}}{\binom{10}{2}} = \frac{2}{9}$.

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Friday, January 29th, 2016
★ Mock Competition ★
Bull's-eye Round
Answer Key & Full Solutions Manual

Form Code

A		F		0		5	
B	■	G		1	■	6	
C		H		2		7	
D		I		3		8	
E		J		4		9	

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Section I: Problem Solving

1. 171

1. 1 Jupiter year equals 6 Mars years. So 15 Jupiter years equals $15 \times 6 = 90$ Mars years.
One Mars year equals 1.9 Earth years. So 90 Mars years equals $90 \times 1.9 = 171$ Earth years.

2. 40

2. Let my starting amount of money in cents be x. After step 1, I have $x - 6 \times 15 = x - 90$ cents left. After step 2, I donate $\frac{1}{5}$ of $x - 90$, leaving me with $\frac{4}{5}(x - 90)$ cents. At this point I've spent half of my money, so $\frac{4}{5}(x - 90) = \frac{1}{2}x$.
Expanding: $\frac{4x}{5} - \frac{90*4}{5} = \frac{x}{2}$
Simplifying: $\frac{4x}{5} - 72 = \frac{x}{2}$
Combining like terms: $\frac{4x}{5} - \frac{x}{2} = 72 \rightarrow \frac{8x}{10} - \frac{5x}{10} = 72 \rightarrow x = 240$
The money I have left after step 2 is 240 divided by 2, or 120 cents. Therefore the scented eraser costs $\frac{120}{3} = 40$ cents each.

3. 04:03 or 4:03

3. The time it takes Alex to get to point B is $\frac{54}{8} = 6.75$ hours. The time it takes Albert to get to point A and back to point B is $\frac{54}{10} \times 2 = 10.8$ hours. The time Alex has to wait is $10.8 - 6.75 = 4.05$ hours, or 4 hours and 3 minutes.

4. 50.4

4. Let x =time travelled upstream (in hours), and y =time traveled downstream (in hours). To maximize the distance travelled, we must have $x+y=6$. The speed of the boat downstream is 28km//hr and the speed of the boat upstream is 12km/hr. Therefore, since the distance travelled downstream is equal to the distance travelled upstream, $28x = 12y$. $x + y = 6 \rightarrow y = 6 - x$, so $28x = 12(6 - x) = 72 - 12x \rightarrow 40x = 72 \rightarrow x = 72/40 = 9/5$. Then $x = 1.8$ and $y = 4.2$, so the maximum distance travelled downstream is $28x = 28(1.8) = 50.4$km.

Section II: Combinatorics and Numbers

5. $\frac{3}{16}$

5. There are a total of 12 marbles. The probability that a white marble is not chosen on the first draw is $1 - \frac{3}{12} = \frac{9}{12} = \frac{3}{4}$. The probability that a white marble is chosen is $\frac{3}{12} = \frac{1}{4}$. The probability that a white marble is first chosen on the second draw is $\frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$

6. 58

6. First, we find the number of positive integers between 1 and 100 that are divisible by 2 or 3. There are 50 such integers that are divisible by 2 and 33 such integers that are divisible by 3. However, we double-counted the multiples of 6, so we subtract 16 from our count (there are 16 integers from 1 to 100 inclusive that are divisible by 6). To find the numbers we already counted that are divisible by 7, we find the number of multiples of 14 or 21 between 1 and 100 inclusive. There are 7 such integers divisible by 14, and 4 such integers divisible by 21. However, now we double-counted the integers divisible by 42, so we subtract the 2 integers between 1 and 100 that are divisible by 42. Our final count is: $50 + 33 - 16 - (7 + 4 - 2) = 58$.

7. 54400

7. Solution 1: Gauss's Trick

Let the sum we are looking for be S. Let the total number of terms that satisfy the conditions be n. Write out all terms twice, in two rows, with the first row starting from the smallest term and increasing, and the second row starting from the largest term and decreasing:

200, 202, 204, 206, 208, 220, 222...

888, 886, 884, 882, 880, 868, 866...

Notice that the difference between two consecutive terms in the top row is equal to the difference between the two consecutive terms right below them in the bottom row. Therefore, if we add the two terms in each column, we always get the same sum: $200 + 888 = 1088$, $202 + 886 = 1088$, and so on. We have n columns that sum to 1088, for a total of $1088n$, but we only want half that, or $540n$, since we wrote out all terms twice. Therefore $S = 540n$. To find n, we need to determine how many combinations of three even digits there are. In the hundreds place we have four possible digits: $2, 4, 6$, or 8. In the tens place, we have five possible digits: $0, 2, 4, 6$, or 8. In the ones place, we again have five possible digits: $0, 2, 4, 6$, or 8. The total number of combinations is $4 \times 5 \times 5 = 100$, so $n = 100$. Finally, we have $S = 540 \times 100 = 54400$.

Solution 2: Smart Digit Summing

We can find S (as defined in Solution 1) by counting all the times each digit shows up and multiplying by the digit's value (100, 10, or 1).

In the ones place, each of 0, 2, 4, 6, and 8 shows up for four different hundreds places and five different tens places, for a total of $4 \times 5 = 20$ times. Therefore the sum of all the ones digits in all the terms that contribute to S is $(0 + 2 + 4 + 6 + 8) \times 20 = 400$.

In the tens place, each of 0, 2, 4, 6, and 8 shows up for four different hundreds places and five different ones places, for a total of $4 \times 5 = 20$ times. Each tens digit must be multiplied by 10 for its true value, so the sum of all the tens digits in all the terms that contribute to S is $(0 + 2 + 4 + 6 + 8) * 20 \times 10 = 4000$.

In the hundreds place, each of 2, 4, 6, and 8 shows up for five different tens places and five different ones places, for a total of $5 \times 5 = 25$ times. Each hundreds digit must be multiplied by 100 for its true value, so the sum of all the hundreds digits in all the terms that contribute to S is $(2 + 4 + 6 + 8) \times 25 \times 100 = 50000$. Again, we have $S = 50000 + 4000 + 400 = 54400$.

8. 71

8. Each of the letters can be swapped with any letters other than the same letter as itself. For example: T cannot be swapped with T. Therefore, if we count the number of possible switches for each of the letters, we get: $10 + 12 + 10 + 11 + 10 + 12 + 12 + 12 + 12 + 10 + 10 + 10 + 11 = 142$ however we double-counted every swap, since swapping X with S is the same as swapping S with X. Therefore, we must divide by 2 to get our final answer: $142 \div 2 = 71$.

Section III: Geometry

9. 864π

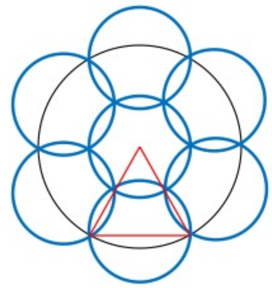
9. The cylinder has radius 12 and height 18. Therefore, the volume would be equal to $12^2 \cdot 18 \cdot \pi = 2592\pi$. The problem statement states that $\frac{2}{3}$ is filled with water and asks for the volume of the unfilled portion. This is simply $\frac{1}{3}$ of the volume: $\frac{2592\pi}{3} = 864\pi$.

10. $2\sqrt{3}$

10. It's clear that this parallelogram is a rhombus as all four of its sides are identical. The diagonal in question can therefore be divided up into two halves, with one half in each equilateral triangle. In fact, each half of the diagonal is actually the height of one of these equilateral triangles. By Pythagorean Theorem we can calculate the height of the equilateral triangle to be $\sqrt{2^2 - 1^2} = \sqrt{3}$. The desired length of the diagonal is therefore $2\sqrt{3}$.

11. 7

11. After experimenting for a little bit, we came up with the following diagram:



It's not hard to prove that these 7 smaller blue circles do indeed fully cover the larger black circle. A sketch of the proof is this: divide up the circle into six equal parts. Then use the properties of $30-60-90$ degree triangles to prove that all of these parts are covered by at least one of the seven circles.

Now we show that 6 or fewer smaller circles cannot fully cover the larger circle. We first note that to fully cover the larger circle, every point on the larger circle's border must be on or inside at least one of the smaller circles. The length of the larger circle's border contained inside one of the smaller circle's boarder depends on the length of the chord in the smaller circle cut off by the intersection points of the two circles. The maximum length of the chord cut off is therefore the diameter the smaller circle, which corresponds to $\frac{1}{6}$ of the larger circle's circumference. Therefore, at least 6 circles are needed to fully cover the larger circle's circumference. However, if only 6 circles are used, the placement of these circles must be that of the 6 outer smaller circles in the diagram above, which obviously leaves the center of the larger circle uncovered. Therefore, at least 7 smaller circles are needed to cover the larger circle. Since we have a construction that works for 7 circles, our answer is 7.

12. $70\sqrt{3}$

12. First, we prove that the largest such hexagon will have one of its sides on the y axis.

It's evident that if we want to maximize the size of such a hexagon, at least one of its vertices must be on the y axis. This is because the entire hexagon must be in the first quadrant, so if an edge of the hexagon intersects the y axis at some point, parts of the hexagon would be outside the first quadrant, and if no part of the hexagon touches the y axis, then the hexagon can be made bigger without having parts of it go into other quadrants (the center $(5, 17)$ is closer to the y axis than the x axis, and since the hexagon is regular, the symmetric properties of the hexagon indicates that a vertex must touch the y axis before another vertex can touch the x axis). Now note if just one vertex is on the y axis, then this vertex can be slid away from $(5, 17)$, rotating the hexagon and therefore increasing its size until an entire edge of the hexagon coincides with the y axis. This shows that the largest such hexagon will have one of its sides on the y axis.

Since the largest such hexagon will have one of its sides on the y axis, the perpendicular bisector of said side must pass through the center of the hexagon. Thus, the length from center of one of the sides to the center of the hexagon is 5. Using this information, the length of one side of the hexagon can be found using the properties of $30 - 60 - 90$ triangles to be $\frac{10\sqrt{3}}{3}$, which gives us the area of $\frac{1}{6}$ of the hexagon (an equilateral triangle) to be $\frac{25\sqrt{3}}{3}$. The area of the hexagon is therefore $50\sqrt{3}$. The perimeter of the hexagon can be also found using the properties $30 - 60 - 90$ triangles: the distance from the center of the hexagon to a vertex is $\frac{10\sqrt{3}}{3}$. Since this distance is equal to the side length of the hexagon, the perimeter of the hexagon is $20\sqrt{3}$. Adding the numerical values of the perimeter and the area together, we get $70\sqrt{3}$.

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Friday, January 29th, 2016
★ Mock Competition ★
Co-op Round
Answer Key & Full Solutions Manual

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A		F		0		5	
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1. 72

1. If Beryl has x dollars, then Alphonse has $2x$ dollars, which gives us the equation $2x + x = 216 \to x = 72$. Alphonse has $2x - x = x$ dollars more than Beryl, or 72 dollars.

2. 11

2. Every time that Gear A completes one full rotation, both gears rotate by 72 teeth. Therefore, both gears must rotate by $(72, 88) = 792$ teeth for the two teeth to touch each other again. This corresponds to $792 \div 72 = 11$ rotations of Gear A.

3. 4

3. If the prime factorization of a number n is $p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ for primes $p_1, p_2, \cdots, p_k$ and positive integers $r_1, r_2, \cdots, r_k$, then the number of divisors of the number is $(r_1 + 1)(r_2 + 1) \cdots (r_k + 1)$. For $(r_1 + 1)(r_2 + 1) \cdots (r_k + 1) = 3$, we note that $r_i + 1 \geq 2$ for all $1 \leq i \leq k$. Therefore, the prime factorization of an n that satisfies $\pi(n) = 3$ can only consist of one prime (two or more primes in the prime factorization indicate there are least four divisors, which isn't allowed). This gives us $r_1 + 1 = 3 \to r_1 = 2$. This shows that the possible integers must be squares of prime numbers. The numbers between 1 and 100 inclusive which are squares of prime numbers are $2^2 = 4, 3^2 = 9, 5^2 = 25$, and $7^2 = 49$. Therefore, 4 possible values exist.

4. $\frac{51}{8}$

4. From the given information, we can say that ΔABC has leg $a = 3$, leg b, and hypotenuse c. ΔDEF has leg $d = 5$, leg $e = c + 3$, and hypotenuse $f = 5 + b$. By the Pythagorean Theorem on both triangles

$$\begin{cases} 3^2 + b^2 = c^2 \\ 5^2 + (c+3)^2 = (5+b)^2 \end{cases}$$

Expanding:

$$\begin{cases} 9 + b^2 = c^2 \\ 25 + c^2 + 6c + 9 = 25 + 10b + b^2 \end{cases}$$

Substituting $9+b^2$ for c^2 in the second equation gives: $25+9+b^2+6c+9 = 25 + 10b + b^2 \to 9 + 3c = 5b \to b = \frac{9+3c}{5}$.
Substituting $b = \frac{9+3c}{5}$ into c^2: $9 + (\frac{9+3c}{5})^2 = c^2 \to 9 + \frac{81+54c+9c^2}{25} = c^2$.
Multiplying everything by 25 and simplifying: $306 + 54c + 9c^2 = 25c^2 \to 16c^2 - 54c - 306 = 0 \to 8c^2 - 27c - 153 = 0$. Factoring gives $(c+3)(8c - 51) = 0$. Since we want the positive solution for c, we must have $c = \frac{51}{8}$.

5. 23

5. It's easy to see that all of the integers from 0 to 10 inclusive can be scores on this contest. Since there are 11 possible scores, we must have at least $2 \times 11 + 1 = 23$ students take the contest in order to guarantee that at least three students will get the same score.

6. 55

6. There are only 8 possible values for n: 2, 3, 5, 7, 11, 13, 17, and 19. For each value of n, we can list the possible values for m, which are all the prime numbers less than $2n$. Therefore, if:
$n = 2, m = \{2, 3\}$,
$n = 3, m = \{2, 3, 5\}$,
$n = 5, m = \{2, 3, 5, 7\}$,
$n = 7, m = \{2, 3, 5, 7, 11, 13\}$,
$n = 11, m = \{2, 3, 5, 7, 11, 13, 17, 19\}$,
$n = 13, m = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$,
$n = 17, m = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31\}$,
$n = 19, m = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\}$,
Summing the number of possibilities for each value of n, we find that there are 55 pairs (m, n) which work.

7. $\frac{11}{21}$

7. There are $\binom{15}{5} = \frac{15!}{5!10!} = 3003$ ways to divide the 15 individuals into a group of 5 and a group of 10. If Anna and Margaret are in the same group of 10, we would need to put the other 13 people into a group of 8 and a group of 5 (the group of 8 would join Anna and Margaret to form a group of 10). Therefore, there are $\binom{13}{8} = \frac{13!}{5!8!} = 1287$ ways for Anna and Margaret to be in the group of 10 together. If Anna and Margaret are in the same group of 5, we would need to put the other 13 people into a group of 10 and a group of 3 (the group of 3 would join Anna and Margaret to form a group of 5). Therefore, there are $\binom{13}{3} = \frac{13!}{10!3!} = 286$ ways for Anna and Margaret to be in the group of 5 together. We sum the two cases to find all the possible ways that Anna and Margaret can end up in the same group, and then divide the result by the total number of ways to divide the 15 people into a group of 5 and a group of 10 to get our answer: $\frac{286+1287}{3003} = \frac{11}{21}$.

8. $5\sqrt{2}$

8. By the symmetry of the figure, we must have $AE = AF$, $\angle EAG = \angle GAF$, and $AG = AG$. By SAS, we therefore have $\Delta AEG \cong \Delta AFG$. This shows that $\angle AGE = 180 \div 2 = 90°$. Since AE is the angle bisector of $\angle BAC$, we have that $\angle BAE = \angle EAC$. Also, we know that $\angle AGE = \angle EBA = 90°$ and $AE = AE$, so by AAS, $\Delta ABE \cong \Delta AEG$. This implies that $AB = AG$. Since we know the square's diagonal AC has length 10, the square's side length must be $AB = \frac{10}{\sqrt{2}} = 5\sqrt{2}$. So $AG = 5\sqrt{2}$.

9. $\frac{9}{19}$

9. Let the total number of celery sticks eaten be C and the total number of hot dogs eaten be H. Since everyone who attended ate exactly one item, the number of people who attended equals $C + H$, and the fraction we are looking for is $\frac{H}{C+H}$.

We know that $\frac{5}{9}H$ hot dogs were eaten by engineers, so the rest, or $\frac{4}{9}H$ hot dogs, were eaten by entomologists. Likewise, $\frac{11}{20}C$ celery sticks were eaten by entomologists, so the rest, $\frac{9}{20}C$ celery sticks, were eaten by engineers. Since an equal number of engineers and entomologists attended, we have:

$$\frac{5}{9}H + \frac{9}{20}C = \frac{4}{9}H + \frac{11}{20}C$$

Simplifying:

$$100H + 81C = 80H + 99C \rightarrow 20H = 18C \rightarrow C = \frac{10H}{9}$$

We substitute $\frac{10H}{9}$ for C in $\frac{H}{C+H}$ to get $\frac{H}{H+\frac{10H}{9}}$, or $\frac{H}{\frac{19H}{9}}$, or $H \times \frac{9}{19H}$. Simplifying the final fraction gives us our answer: $\frac{9}{19}$.

10. $\frac{\sqrt{2}+\sqrt{6}}{2}$

10. Suppose $\sqrt{2+\sqrt{3}} = \frac{\sqrt{a}+\sqrt{b}}{c}$ for the a, b, c defined in the problem statement. Squaring both sides gives $2+\sqrt{3} = \frac{a+b+2\sqrt{ab}}{c^2}$. Simplifying $2+\sqrt{3} = \frac{a+b+2\sqrt{ab}}{c^2}$ gives $2c^2 + c^2\sqrt{3} = a+b+2\sqrt{ab}$. Since a, b, c are integers, we must have $2c^2 = a+b$ and $c^2\sqrt{3} = 2\sqrt{ab}$. Manipulating $c^2\sqrt{3} = 2\sqrt{ab}$ gives $\sqrt{3c^4} = \sqrt{4ab} \to 3c^4 = 4ab$. So we have the system:

$$\begin{cases} 2c^2 = a+b \\ 3c^4 = 4ab \end{cases}$$

From the first equation, we have $c^2 = \frac{a+b}{2}$. Plugging this into the second equation gives $3\left(\frac{a+b}{2}\right)^2 = 4ab \to 3a^2+6ab+3b^2 = 16ab \to 3a^2 - 10ab + 3b^2 = 0$. Now we factor: $(a-3b)(3a-b) = 0$. So either $a = 3b$ or $a = \frac{b}{3}$. For $a = 3b$, we have $2c^2 = 3b+b = 4b \to c^2 = 2b$. If c is odd, c^2 is odd so b cannot be an integer, which isn't allowed. If c is negative , then $\frac{\sqrt{a}+\sqrt{b}}{c} < 0$ which clearly isn't the case for $\sqrt{2+\sqrt{3}}$. So if c is positive, even, and greater than 2, then $c^2 = (2m)^2$ for some positive integer $m > 1$. This means that $(2m)^2 = 2b \to b = 2m^2$ which implies that b is divisible by the square of an integer greater than 1, which according to the problem statement, isn't allowed. So $c = 2$, which means $b = 2$. This gives $a = 6$. For $a = \frac{b}{3}$, we have $2c^2 = \frac{4b}{3} \to 3c^2 = 2b$. By a very similar argument as in the previous case, we can show that $c = 2$ in this case, yielding $b = 6$. Therefore, $a = 2$. In both cases, our answer is $\frac{\sqrt{2}+\sqrt{6}}{2}$.

11. 7

11. First notice that $250047 = 3^6 \cdot 7^3$. We can factor $x^6 - 3x^4y^2 + 3x^2y^4 - y^6$ into $(x^2-y^2)^3 = ((x-y)(x+y))^3$, which gives us $((x-y)(x+y))^3 = 250047$. Cube rooting both sides, we get $(x-y)(x+y) = 63$. Since we have the bound that x and y are both positive integers smaller than 9, the only possible pair for x and y is $x = 8$ and $y = 1$. This yields the answer: $\frac{x^2-y^2}{x+y} = x - y = 8 - 1 = 7$.

12. 36

12. There are four possible remainders when a number is divided by 4: 0, 1, 2, and 3. Therefore, exactly one of the four chosen integers must have a remainder of 0 when divided by 4. The same is true for the other three possible remainders, 1, 2, and 3. There are 2 integers from 1 to 10 inclusive which give a remainder of 0 when divided by 4. There are 3 integers from 1 to 10 inclusive which give a remainder of 1 when divided by 4. There are 3 integers from 1 to 10 inclusive which give a remainder of 2 when divided by 4. There are 2 integers from 1 to 10 inclusive which give a remainder of 3 when divided by 4. Therefore, there are in total $2 \times 3 \times 3 \times 2 = 36$ ways to pick these four integers.

13. -15

13. Solution 1: Algebra

Suppose the next vertex going clockwise from $(4,7)$ has coordinates (x,y). Using the distance formula, $(7-y)^2+(4-x)^2=(2\sqrt{65})^2=260$. The equation of the line that these two vertices lie on can be modelled using the point-slope form equation $7-y=\frac{-4}{7}(4-x)$. We notice that both $7-y$ and $4-x$ show up in the previous two equations, so we substitute them with a and b respectively. Our new equations are therefore:

$$\begin{cases} a^2+b^2=260 \\ a=\frac{-4}{7}b \end{cases}$$

Substituting the second equation into the first gives $\left(\frac{-4}{7}b\right)^2+b^2=260 \rightarrow \frac{65b^2}{49}=260 \rightarrow b=\pm 14$. Plugging back into $a=\frac{-4}{7}b$ gives $a=\mp 8$. For $b=14$ and $a=-8$, we have $7-y=-8$ and $4-x=14$ which gives us $(x,y)=(-10,15)$. For $b=-14$ and $a=8$, we have $7-y=8$ and $4-x=-14$ which gives us $(x,y)=(18,-1)$. Because this vertex has to be in the fourth quadrant, we can reject $(x,y)=(-10,15)$ as a possible set of coordinates and instead take $(x,y)=(18,-1)$ as our coordinate. Now, we know that all sides of a square are equal in length and the slopes of the lines that any two of its sides lie on are either the same or are negative reciprocals of each other. Since the vertex $(18,-1)$ is 8 units down and 14 units to the right of the vertex $(4,7)$, the next vertex going clockwise must be 14 units down and 8 units to the left of $(18,-1)$, at $(10,-15)$. The final vertex is therefore 8 units up and 14 units to the left of $(10,-15)$, at $(-4,-7)$. Therefore, our smallest y-coordinate is -15.

Solution 2: Shortcuts

Since the slopes of perpendicular lines are negative reciprocals of each other and two sides of the square lie on lines with slope $\frac{-4}{7}$, the other sides must have slopes $\frac{7}{4}$. This means one of the sides must pass through the origin: since our known vertex is at $(4,7)$ If we move 4 units to the left and 7 units down to $(0,0)$, we can use the distance formula to find the distance from $(4,7)$ to $(0,0)$ equals as $\sqrt{4^2+7^2}=\sqrt{65}$. Our square's side length is $2\sqrt{65}$, so we simply move 4 units to the left and 7 units down again to double the length, bringing us to the vertex in quadrant III, $(-4,-7)$. To get to the next vertex, we do the same, except we move 4 units down and 7 units to the right **twice** since the slope is now $\frac{-4}{7}$. This brings us to the point $(10,-15)$. Clearly this vertex has the smallest y-coordinate of the four vertices, so our answer is -15.

14. 13×3^{28} or $13^1 \times 3^{28}$

14. Suppose the first term is a and the common ratio is x^2 for some integers a and x (we know $x>1$ from the problem statement), we have the fifth term as ax^8. Since 85293 can be prime factorized into $13 \cdot 3^8$, a must be 13 and x be 3. So the 29$^{\text{th}}$ term is $a \cdot x^{56}$ and the first term is a. Their product is $a^2 \cdot x^{56}$. The square root of this product is $n \cdot x^{28} = 13^1 \times 3^{28}$.

15. 77

15. $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. The positive divisors of 120, from smallest to largest, are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120. The sum of these divisors is 360. The height of the trapezoid is 5, so the sum of the lengths of the two parallel sides of the trapezoid is 144.

Suppose the shorter of the two parallel sides of the trapezoid has length n. Since one of the legs lie on a line with slope $\frac{5}{3}$, we know that the other leg lies on a line with slope $-\frac{5}{3}$. Since the two parallel sides are parallel to the x axis and the trapezoid is isosceles, we know that the longer parallel side of the trapezoid has length $2 \times 3 + n = 6 + n$ (the height of the trapezoid is 5). Adding the lengths of both parallel sides, we get $144 = 6 + 2n$ which gives us $n = 69$. To maximize the sum of the x and y coordinates of one of the vertices of this trapezoid, we must have the four vertices be at $(0, 0)$, $(-3, 5)$, $(72, 5)$, and $(69, 0)$. The maximum sum desired is therefore $72 + 5 = 77$.